From particle in a box to $\mathcal{PT}$-symmetric systems via isospectral deformation

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A family of $\mathcal{PT}$-symmetric complex potentials are obtained which is isospectral to free particle in an infinite complex box in one dimension (1-D). These are generalizations to the $cosec^2(x)$ potential, isospectral to particle in a real infinite box. In the complex plane, the infinite box is extended parallel to the real axis having a real width, which is found to be an integral multiple of a constant quantum factor, arising due to boundary conditions necessary for maintaining the $\mathcal{PT}$-symmetry of the superpartner. As the spectra of the particle in a box is still real, it necessarily picks out the unbroken $\mathcal{PT}$-sector of its superpartner, thereby invoking a close relation between $\mathcal{PT}$-symmetry and SUSY for this case.

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Introduction

Supersymmetric quantum mechanics [1, 2] provides an elegant connection between two different systems with Hamiltonians in factorized form [3]. The additional property of shape-invariance [4] ensures solvability of the potential by relating it to a family of systems with successive isospectrality [1]. Interesting phenomena, like reflectionlessness potentials [5] can be explained to be due to isospectrality of the system [2] with free particle. On the other hand, $\mathcal{PT}$-symmetric systems [6] have been under consideration owing to the realness of the spectra belonging to non-Hermitian complex Hamiltonians (potentials). These turn into complex-conjugate pairs of energy for a different range of parameters interpreted as spontaneous breaking of $\mathcal{PT}$-symmetry. For unbroken $\mathcal{PT}$-symmetry, the realness of the spectra necessarily demands generalization of the usual Dirac-von Neumann scalar product [7], generically unique to the particular system, in the same spirit of the pseudo-Hermitian systems. It should be emphasized that the equivalence of $\mathcal{PT}$-symmetry and pseudo-Hermiticity is yet to be established [8]. Certain class of $\mathcal{PT}$-symmetric potentials display isospectrality to Hermitian systems [9]. This is expected to hold in the *same* parameter range, where $\mathcal{PT}$-symmetry is unbroken. The necessary realness of both spectra is ensured by supersymmetry. Thus, a co-existence of both the symmetries is observed. Recently, supersymmetric structure of certain $\mathcal{PT}$-symmetric systems has been considered to analyze the structure of the corresponding Hilbert spaces, along with the parametric correspondence between the two [10]. Subsequently, scattering properties of $\mathcal{PT}$-symmetric systems were qualitatively studied to realize unique boundary conditions arising for such systems [11], in accord to the experimental results [12].

In this paper, we will analyze and generalize the simplest quantum mechanical system, the one dimensional infinite well, to generate a whole family of $\mathcal{PT}$-symmetric potentials through isospectral deformation [2]. Strict boundary conditions necessarily require complexification of the 1-D space, thereby resulting in multiple choices for the parity transformation. We stick to the definition of parity that alters the sign of only the real part of the co-ordinate, in order to stay close to physical realization. It follows that the width of the well, which has to be real for the realness of the physical spectrum, cannot be arbitrary. Instead, it has to be an integral multiple of a constant, manifestly quantum in nature. $\mathcal{PT}$-symmetric systems, under constant *imaginary* shift of coordinate, have been studied [13] for their supposed 'η-pseudo-Hermiticity' [14]. Hhere η is the proposed 'norm operator' for the pseudo-Hermitian systems [15]. In the present case, we utilize 'general isospectral deformation' to arrive at *complex* shift in coordinate of the 'real' system, obtaining a map to a $\mathcal{PT}$-symmetric system, defined in real space with imaginary coordinate acting as a continuous parameter, subjected to the physical boundary conditions. A scheme of extending the isospectral family of potentials to include non-Hermitian systems has been realized.

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The paper has been organized as follows. Section I contains a brief summary of isospectral deformation leading to the solutions of the Bernoulli's equation. Section II deals with the family of equivalent superpotentials corresponding to 1-D particle in an infinite box and that the resulting isospectral family of potentials necessarily require complexification of space and discretization of the well width. Section III contains a generalized treatment of 1-D infinite well in complex plane, emphasizing on the importance of boundary conditions necessary for physical results. Section IV analyzes the $\mathcal{PT}$-symmetric potentials, isospectrally constructed from the generalized particle in a box, and also the parametric conditions leading to spontaneous breaking of $\mathcal{PT}$-symmetry, thus shading light on its effect on the isospectrality. In conclusion, the accumulated understanding of these generalized systems will be discussed, along with the prospects for future works.

I. ISOSPECTRAL DEFORMATION

Given a potential leading to a factorizable Hamiltonian, the corresponding superpotential is not unique. However, different superpotentials correspond to different isospectral partners of the unchanged potential in general. Such modification of the superpotential is known as the isospectral deformation, and evidently in this way, a family of superpartners can be constructed for a given potential. An isospectral pair of potential is conventionally written as,

$$\begin{aligned} V_-(x) &= W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) \\ V_+(x) &= W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x), \end{aligned} \tag{1}$$

which have exactly the same energy eigenvalues, except for the ground energy of $V_-(x)$ with no mapping into $V_+(x)$ [*i.e.* $E_n^+ = E_{n+1}^-$], but different eigenfunctions which are algebraically related as [1],

$$\psi_n^+(x) = [E_{n+1}^-]^{-\frac{1}{2}} K \psi_{n+1}^- \quad \text{or} \quad \psi_{n+1}^-(x) = [E_n^+]^{-\frac{1}{2}} K^\dagger \psi_n^- .$$

Here, $W(x)$ is the superpotential and $W'(x)$ is its first derivative with respect to x. We invoke the *isospectral deformation* $W(x) \longrightarrow W(x) + g(x)$, $g(x)$ being an arbitrary function restricted only by the boundary condition of the unchanged system, chosen for convenience to be that for $V_-(x)$, and normalizability of the corresponding wavefunctions for the *new* $V_+(x)$. Then, from Eq.1, for unchanged $V_-(x)$ we obtain:

$$g^2(x) + 2W(x)g(x) - \frac{\hbar}{\sqrt{2m}} g'(x) = 0, \tag{2}$$

with a new superpartner,

$$V_+(x) = (W(x) + g(x))^2 + \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} (W(x) + g(x)) . \tag{3}$$

The physically acceptable solution $g(x)$ of the Bernoulli's equation (Eq.2), for a given $W(x)$, is determined by the normalizability of the eigenfunctions corresponding to $V_+(x)$.

II. ISOSPECTRALITY OF THE FREE PARTICLE IN INFINITE BOX

The free particle corresponds to a constant superpotential $W(x) = A$. Presence of local boundary conditions generally modifies it to a local function, as in the case of 1-D infinite box, reflected in the expression of the ground state in terms of the superpotential, $\psi_0(x) = exp\left[-\int^x W(x')dx'\right]$. The constant potential A^2 can always be absorbed in defining the origin of energy. The corresponding Bernoulli's equation reads as,

$$g^2(x) + 2Ag(x) - \frac{\hbar}{\sqrt{2m}} g'(x) = 0,$$

with the solution,

$$g(x) = -2A\frac{\mathbf{e}^{2A(\frac{\sqrt{2m}x}{\hbar}+c)}}{\mathbf{e}^{2A(\frac{\sqrt{2m}x}{\hbar}+c)}-1}, \tag{4}$$

with c being the constant of integration. The fact that $V_-(x)$ is a constant, leads to the Riccati equation: $W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) = \textit{constant}$ for the superpotential, having unique solution modulo an additive constant. Though it is trivial in this particular case, it ensures uniqueness of $g(x)$ too. The new superpotential $-A\coth\left(A\frac{\sqrt{2m}}{\hbar}x + Ac\right)$, which will hereon be called $W(x)$, generically marks the presence of local boundaries and leads to a *new* superpartner,

$$V_+(x) = 2A^2\sinh^{-2}\left(A\frac{\sqrt{2m}}{\hbar}x + Ac\right) + A^2. \tag{5}$$

It is to be emphasized that the functional form of the potential ($V_-(x)$) by itself does not contain information of the boundary conditions, but the superpotential does, as it determines the ground sate eigenfunction. Thus, through isospectral deformation, only the value of $V_-(x)$ is unchanged, not the system boundaries.

The ground state wave function for $V_-(x)$ in terms of the new superpotential is,

$$\psi_0^-(x) = N\left(-\mathbf{e}^{-A\frac{\sqrt{2m}}{\hbar}x} + \mathbf{e}^{A\frac{\sqrt{2m}}{\hbar}x+2Ac}\right), \tag{6}$$

which must be same as that for $W(x) = A$, *i.e.*, sinusoidal with nodes only at the boundaries. This necessarily requires that $A = \mathbf{i}a$, where $a\epsilon\mathbb{R}$, yielding,

$$V_+(x) = 2a^2\csc^2\left(a\frac{\sqrt{2m}}{\hbar}x + ac\right) - a^2. \tag{7}$$

The knowledge of the ground state of $V_-(x)$ along with Eq.6 ensures the trivial physical choice of $c = 0$. This is the previously mentioned boundary value restriction from a potential ($V_-(x)$) to its superpartner ($V_+(x)$), as evident from Eq.7. However, we are motivated here to construct the most general form of $V_+(x)$, thus to generalize $V_-(x)$ if necessary. Keeping this in mind, we consider the integration constant c to be complex in general. Then, subjected to the boundary conditions that the eigenfunction vanishes at $x_{1,2}$, one obtains from Eq.6,

$$\mathbf{e}^{2\mathbf{i}ac} = \mathbf{e}^{-2\mathbf{i}a\frac{\sqrt{2m}}{\hbar}x_{1,2}}, \tag{8}$$

thereby necessitating $x_{1,2}$ to be complex. The above equation also ensures the difference between x_1 and x_2 to be $j\frac{\hbar}{\sqrt{2m}}\frac{\pi}{a}$, with j being an integer. Therefore, we end up generalizing the 1-D infinite potential well with arbitrary width to be an infinite 1-D potential well in the complex x plane, placed parallel to the real axes, with *real* but integral 'quantum' width. That this choice is the most general one, is explicated in the appendix. Thus, $V_-(x)$ can correspond to different infinite wells subjected to different combinations of j and a. The eigenvalues, however, are still real as they depend only on $x_1 - x_2$, which is real, but can have different values now. Hence, we ended up isospectrally relating a *family* of 1-D infinite wells to a family of complex potentials, which will be discussed next.

For mathematical and intuitive convenience, we redefine the variable(s) as,

$$\begin{aligned} p &= a\frac{\sqrt{2m}}{\hbar}x_{re} + ac_{re} \\ q &= a\frac{\sqrt{2m}}{\hbar}x_{im} + ac_{im}, \quad \text{with} \quad x = x_{re} + \mathbf{i}x_{im} \end{aligned} \tag{9}$$

Then, one can re-write $x = \frac{a\hbar}{\sqrt{2m}}\left[(p - ac_{re}) + i(q - ac_{im})\right]$ in terms of the new variables. Now, the potential in Eq.7 can be written as $V_+(x) = 2a^2 cosec^2(p + \mathbf{i}q) - a^2$ which can be re-organized to become,

$$V_+(x) = a^2\left[8\frac{\sin^2 p\cosh^2 q - \cos^2 p\sinh^2 q}{(\cos(2p) - \cosh(2q))^2} - \mathbf{i}4\frac{\sin(2p)\sinh(2q)}{(\cos(2p) - \cosh(2q))^2} - 1\right], \tag{10}$$

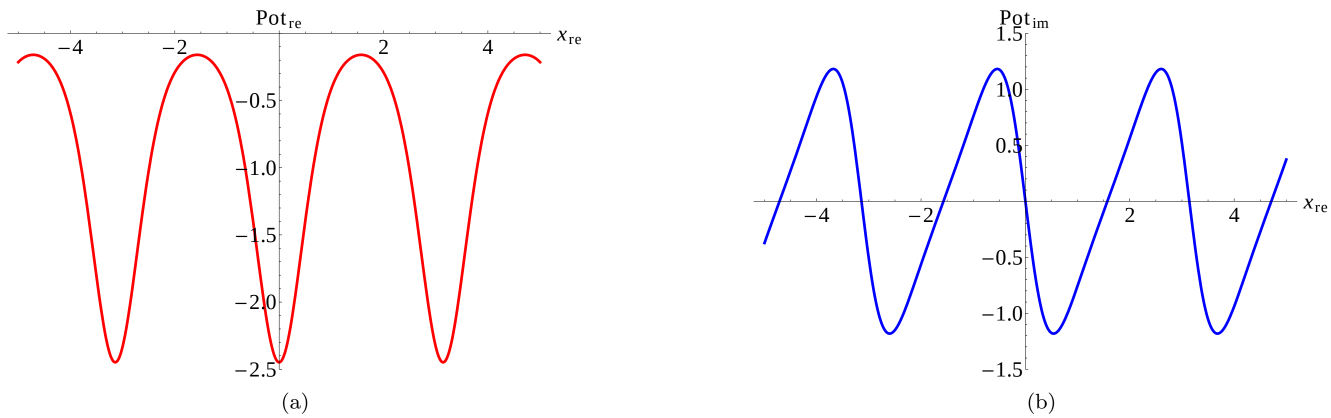

FIG. 1: Plots of (a) $\mathbb{R}e(V_+(x))$ and (b) $\mathbb{I}m(V_+(x))$ vs. x_{re} modulo an additive constant c_{re} (or p).

with real and imaginary parts well-separated. Fig 1 shows the plots of $V_+^{re}(x)$ and $V_+^{im}(x)$ as a function of x_{re}, showing the former to be even and the later to be odd in x_{re}. This ensures $\mathcal{PT}$-symmetry of $V_+(x)$, provided we define parity transformation in the complex x-plane to be $x_{re} \to -x_{re}, x_{im} \to x_{im}$. This is sensible also from the point of view that the *generalized* potential wells are actually manifested through the shift of the origin, as far as the invoked complexity of the coordinate x is concerned. The variable x_{im} acts as a continuous parameter extending from $+\infty$ to $-\infty$, same as a and c, while the variable x_{re} is restricted by boundary conditions for the well.

Hamiltonians in complex coordinate plane have been extensively studied in classical domain [16], wherein multiple trajectories, along with the shifted 'usual' real space trajectories (straight lines), correspond to the *same* probability in corresponding quantum systems [17]. These have different sets of 'classical turning points' with the possibility of 'classical tunneling' [18]. Here, we stick to the shifted 'usual' trajectory for the infinite well, which is fixed parallel to the x_{re} axis by the physical boundary conditions to be explicated in the next section. Subsequently, $V_+(x)$ is confined in real space only, with x_{im} serving as a continuous parameter.

III. THE GENERAL 1-D INFINITE BOX IN COMPLEX SPACE

In order to obtain an isospectral map to $\mathcal{PT}$-symmetric system of $V_+(x)$ in real space, by generalizing a Hamiltonian system (the 1-D infinite box) to the complex plane, we need x_{im} to be at best a continuous parameter, whereas x_{re} is the direction along which parity transformation has to be executed. Here, one expects the motion of the particle in the infinite well to remain confined to the real axis, for the spectra of the system to be real, requiring the *quantized* widths of the well to be real. This requirement makes the treatment fundamentally different from complex canonical transformation in the *phase* space, utilized to study $\mathcal{PT}$-symmetric systems earlier [19], obtaining a 'new' Hamiltonian with real spectra. Here, we do a position-coordinate or 'point' transformation by taking the Hermitian system to complex plane, and then obtain $\mathcal{PT}$-symmetric systems from there through isospectral deformation.

We begin by considering the 1-D infinite box to be oriented in the complex plane $[x_{re}, x_{im}]$ in arbitrary direction instead of parallel to the x_{re}-axis. Consequently, the generalized boundary conditions for the eigenfunction will be $\psi_n(x_{re}^{(1)}, x_{im}^{(1)}) = 0 = \psi_n(x_{re}^{(2)}, x_{im}^{(2)})$, with coordinates in the bracket representing the infinite boundary. Now, the 2-D Laplacian, $\nabla^2 = \frac{\partial^2}{\partial x_{re}^2} - \frac{\partial^2}{\partial x_{im}^2}$ is defined in the complex coordinate plane, yielding the Schrödinger eigenvalue equation,

$$\left(\frac{\partial^2}{\partial x_{re}^2} - \frac{\partial^2}{\partial x_{im}^2}\right)\psi(x_{re}, x_{im}) = -K^2\psi(x_{re}, x_{im}), \quad \text{where} \quad K^2 = \frac{2mE}{\hbar^2}. \tag{11}$$

On separation of variables as $\psi(x_{re}, x_{im}) = \mathcal{R}(x_{re})\mathcal{I}(x_{im})$, one obtains,

$$\frac{1}{\mathcal{R}}\mathcal{R}'' + K^2 = \tilde{K}^2 = \frac{1}{\mathcal{I}}\mathcal{I}'', \tag{12}$$

where $\tilde{K}^2$ is a constant and the prime denotes derivative with respect to the respective variables. The individual solutions are:

$$\mathcal{R}(x_{re}) = A\exp(\mathbf{i}\bar{k}x_{re}) + B\exp(-\mathbf{i}\bar{k}x_{re}), \quad \text{and} \quad \mathcal{I}(x_{im}) = C\exp(\tilde{k}x_{im}) + D\exp(-\tilde{k}x_{im}), \tag{13}$$

Where A, B, C and D are complex numbers and $\bar{K}^2 = K^2 - \tilde{K}^2$. We identify that $\bar{K} = K_{re}$ and $\tilde{K} = K_{im}$, the momenta along respective axes.

The second one of Eq.12 is what one obtains considering a 1-D free particle in imaginary space. In that case, $\tilde{K}$ is *real* as the space is considered to be imaginary. If one considers the variable of differentiation to be $\mathbf{i}x_{im}$, instead of x_{im}, we end up with imaginary momentum. These are two equivalent pictures of a 1-D free particle in imaginary space, which can also be seen as a 1-D particle moving in a constant potential $V > E$, E being the total energy of the particle, known to have exponentially decaying solutions.

On plugging in the boundary conditions $\psi_n(x_{re}^{(1)}, x_{im}^{(1)}) = 0 = \psi_n(x_{re}^{(2)}, x_{im}^{(2)})$ in Eq. 13, we obtain,

$$\begin{aligned} A\exp(\mathbf{i}\bar{k}x_{re}^{(1)}) + B\exp(-\mathbf{i}\bar{k}x_{re}^{(1)}) = 0 = A\exp(\mathbf{i}\bar{k}x_{re}^{(2)}) + B\exp(-\mathbf{i}\bar{k}x_{re}^{(2)}) \\ \text{and} \quad C\exp(\tilde{k}x_{im}^{(1)}) + D\exp(-\tilde{k}x_{im}^{(1)}) = 0 = C\exp(\tilde{k}x_{im}^{(2)}) + D\exp(-\tilde{k}x_{im}^{(2)}), \end{aligned} \tag{14}$$

yielding,

$$\exp\left\{\mathbf{i}\bar{K}(x_{re}^{(1)} - x_{re}^{(2)})\right\} = 1 = \exp(\mathbf{i}2n\pi) \quad \text{and} \quad \exp\left\{\tilde{K}(x_{im}^{(1)} - x_{im}^{(2)})\right\} = 1 = \exp(\mathbf{i}2m\pi), \quad n, m\epsilon\mathbb{I}. \tag{15}$$

As $\bar{K}$, $\tilde{K}$, x_{re} and x_{im} are all real, the only sensible conclusion is $m = 0$, implying $x_{im}^{(1)} = x_{im}^{(2)}$. Thus the box is parallel to the real axis and the particle is constrained to move in 1-D. The identification $\tilde{K} = K_{im}$, owing to the explicit consideration $x_{im}\epsilon\mathbb{R}$, ensures the 'physicality' of the Hermitian system (1-D infinite well) that it has real spectra with the wave-function decaying exponentially along x_{im}, with proper choice of coefficients C and D in Eq.13. This is a general result, owing to both the imaginary nature of one variable and the infinite boundary.

The above problem can equivalently be cast as one in the $SO(1,1)$ space (x, y) (the $1+1$ Minkowski space), where the conserved scalar is $x^2 - y^2$. The corresponding conserved scalar in the Fourier space is $K_x^2 - K_y^2$, with the metric $g_{\mu\nu} = \text{Diag}(1, -1)$ or $g_{\mu\nu} = \text{Diag}(1, \mathbf{i})$, where the space is essentially considered to be real. We adopted the notation of the equivalent metric $g_{\mu\nu} = (1, 1)$ $(SO(2))$, with explicit consideration that the space is complex. This leads to the conserved scalar $x^2 - y^2$ same as that in the $1+1$ Minkowski space, but now that in the Fourier/momentum space is $K_x^2 + K_y^2$. Thus under the $SO(2)$ metric, either space (1-D component) can be real or the corresponding momenta, but not both, if the complete space is complex. The essence to carry forward here is that in the present problem, the complexness of space is not avoidable as the $SO(2)$ metric is unique, and hence the imaginary component of the space is *not* an artifact, removable by choice of metric, unlike the $SO(1,1)$ case.

IV. THE GENERALIZED FAMILY OF $\mathcal{PT}$-SYMMETRIC POTENTIALS

The spectrum of $V_-(x)$ is still real, which can be verified in the usual manner by considering the general plane wave solution $A\mathbf{e}^{\mathbf{i}kx} + B\mathbf{e}^{-\mathbf{i}kx}$ to vanish at complex values $x_{1,2}$, leading to $k(x_2 - x_1) = n\pi$ with $(x_2 - x_1) = L$ being real. This, through isospectrality, necessarily demands the spectrum of $V_+(x)$ to be real. But being $\mathcal{PT}$-symmetric, $V_+(x)$ also has complex-conjugate spectra when the $\mathcal{PT}$-symmetry breaks spontaneously. Thus the isospectrality with the complex infinite well should persist only till the $\mathcal{PT}$-symmetry is preserved. To analyze the complete spectra of $V_+(x)$, we seek to solve the eigenvalue problem for $H_+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_+(x)$. For convenience, we consider the form in Eq.7, leading to the eigenfunctions [1]:

$$\psi_n^+(x) \propto (n+1)\cos\left\{(n+2)(a\frac{\sqrt{2m}}{\hbar}x + ac)\right\} - \sin\left\{(n+1)(a\frac{\sqrt{2m}}{\hbar}x + ac)\right\}\csc\left(a\frac{\sqrt{2m}}{\hbar}x + ac\right), \tag{16}$$

modulo a normalization constant. On considering Eq. 9, the above equation can be rewritten as:

$$\begin{aligned} \psi_n^+(x) &\propto (n+1)\cos\{(n+2)(p+\mathbf{i}q)\} - \sin\{(n+1)(p+\mathbf{i}q)\}\csc(p+\mathbf{i}q) \\ &\propto (n+1)\left[\cos\{(n+2)p\}\cosh\{(n+2)q\} - \mathbf{i}\sin\{(n+2)p\}\sinh\{(n+2)q\}\right] \\ &- \left[\sin\{(n+1)p\}\cosh\{(n+1)q\} + \mathbf{i}\cos\{(n+1)p\}\sinh\{(n+1)q\}\right]\left[\sin(p)\cosh(q) - \mathbf{i}\cos(p)\sinh(q)\right]^{-1} \end{aligned} \tag{17}$$

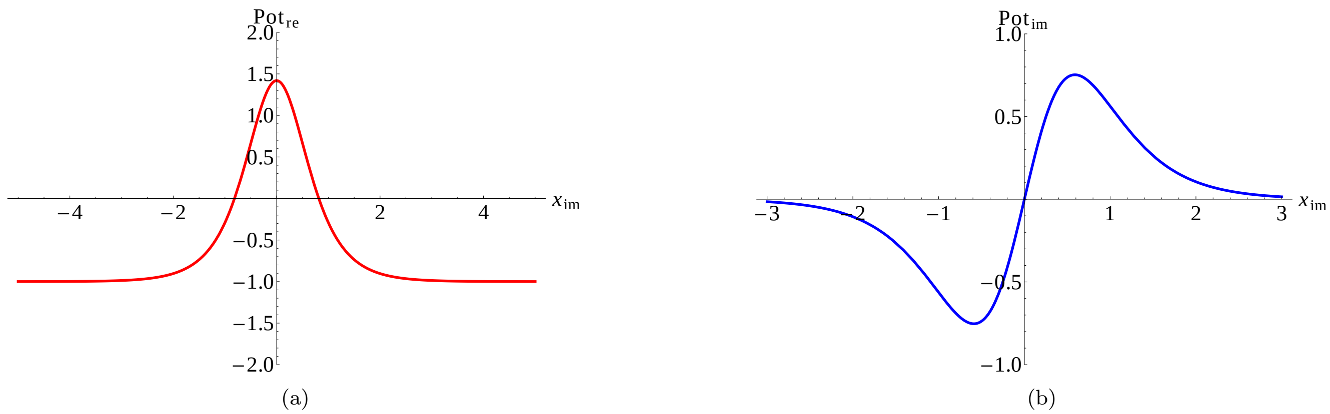

FIG. 2: Plots of (a) $\mathbb{R}e(V_+(x))$ and (b) $\mathbb{I}m(V_+(x))$ vs. x_{im} modulo an additive constant c_{im} (or q).

Clearly, these eigenfunctions are $\mathcal{PT}$-symmetric with respect to x_{re} modulo a shift ac_{re} of the origin. This is in accord with the property of isospectral $\mathcal{PT}$-symmetric potentials that unique superpotential leads to unbroken $\mathcal{PT}$-symmetric sector, as observed in [10], unless any further symmetry is involved [20].

From Eqs 10 and 17, it is also evident that both $V_+(x)$ and $\psi_n^+(x)$ remain $\mathcal{PT}$-symmetric subjected to the alternative choice of parity transformation: $x_{re} \to x_{re}, x_{im} \to -x_{im}$, with x_{re} acting as a continuous parameter (Fig 2). However, this will not correspond to a particle in an infinite box directly, as such a system cannot exist parallel to the imaginary axis. Therefore, no alteration of the parameters can lead to a spontaneously broken $\mathcal{PT}$-symmetric regime at this stage. Naturally, there exists a more general form of $V_+(x)$, which reduces to that in Eq.10 under certain parametric condition, preserving $\mathcal{PT}$-symmetry spontaneously [10]. To obtain the same, instead of $-A\coth\left(A\frac{\sqrt{2m}}{\hbar}x + Ac\right)$, we propose a superpotential:

$$W(x) = -a\cot\left(\alpha\frac{\sqrt{2m}}{\hbar}x + \alpha c\right) + \mathbf{i}B, \quad a = -\mathbf{i}A \;\text{ and }\; \alpha\epsilon\mathbb{R}, B\epsilon\mathbb{Z}, \tag{18}$$

leading to,

$$V_+(x) = \left(a^2 + \alpha a\right)\csc^2\left(\alpha\frac{\sqrt{2m}}{\hbar}x + \alpha c\right) - \mathbf{i}2aB\cot\left(\alpha\frac{\sqrt{2m}}{\hbar}x + \alpha c\right) - \left(a^2 + B^2\right). \tag{19}$$

The potential in Eq.7 corresponds to a particular choice $\alpha = a$ and $B = 0$. For $\alpha \neq a$ and $B = 0$, the system will not map to a free particle (in or out of the box!) as $V_-(x) = \left(a^2 - \alpha a\right)\csc^2\left(\alpha\frac{\sqrt{2m}}{\hbar}x + \alpha c\right) - a^2$ with ground state wave function $\psi_0^-(x) \propto \left\{\sin\left(\alpha\frac{\sqrt{2m}}{\hbar}x + \alpha c\right)\right\}^{-a/\alpha}$. It is evident that $V_\pm(x)$ are shape-invariant [1, 2, 4] and hence, can be solved exactly using the SUSY algebra. The spectrum turns out to be $E_n = a^2 - (a - n\alpha)^2$ [1, 2], which is real. Therefore, the $\mathcal{PT}$-symmetry is still preserved. On the other hand, for $\alpha \neq a$ and $B \neq 0$, it is known [10] that there exists a spontaneously broken $\mathcal{PT}$-symmetric phase, with complex-conjugate pairs of discrete, yet finite eigenvalues, even for $x\epsilon\mathbb{R}$.

Therefore, in the complex space, the free particle in infinite 1-D box isospectrally maps to a restricted portion ($\alpha = a,\ B = 0$) of the parametric domain for spontaneously preserved $\mathcal{PT}$-symmetry of the potential in Eq.19. Such results were known earlier [21] for specific $\mathcal{PT}$-symmetric potentials obeying SUSY-QM. Isospectral deformation plays a distinct role in the present case, by bringing out the most general form of the superpotential corresponding to a constant potential. Generic complexification yielded the $\mathcal{PT}$-symmetric potential, isospectral to the free particle in a *real* 1-D infinite box, embedded in the complex space.

Conclusions

The fact that the real spectrum of a $\mathcal{PT}$-symmetric potential can be mapped to that of a much simpler potential through suitable generalization of the space enables us to intuitively picture why the spectrum of such a potential is

real, even without Hermiticity, due to isospectrality to a Hermitian system. Here, we have extended the domain of such analogy through generic complexification of the space. This might be helpful to visualize the real spectrum of other $\mathcal{PT}$-symmetric potentials. This could serve as an alternative to constructing a new scalar product to explain the real spectrum of these complex potentials, by attributing the realness of the spectrum to parametric isospectrality.

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